Split-merge dynamics for the Voronoï cells of the circle

proposed by Serge Cohen office 229 bat 1R1
tel 05 61 55 85 74
Serge.Cohen@math.univ-toulouse.fr

In a recent paper https://arxiv.org/abs/2502.09114 we study sequences of partitions of the unit interval into subintervals, starting from the trivial partition, in which each partition is obtained from the one before by splitting its subintervals in two, according to a given rule, and then merging pairs of subintervals at the break points of the old partition. The nth partition then comprises $n+1$ subintervals with n break points, which inherently possess an interlacing property. The empirical distribution of these points reveals a surprisingly rich structure. Extension to split-merge dynamics of the circle have been considered during Rossmann's passed internship. A goal of the present internship will be to study in depth this article and Rossmann's work.

Here we propose to consider split-merge dynamics for the Voronoï cells of the circle. Fix a positive integer N and let E_N denote the set of subsets of the circle $S = \mathbb{R}/\mathbb{Z}$ of size N. For $x \in E_N$, we will use coordinates $(x_1, \dots, x_N)$ where $x_1 \dots, x_N \in (0, 1]$ with $x_1 < \cdots < x_N$ and $x = \{x_1 + \mathbb{Z}, \dots, x_N + \mathbb{Z}\}$. We write $x = (x_1, \dots, x_N)$ and we set $x_0 = x_N - 1,\ x_{N+1} = x_1 + 1$. For $x, y \in E_N$, we say that x and y are interlaced if one of the following hold:

$$\text{either} \quad x_1 < y_1 < \cdots < x_N < y_N \quad \text{or} \quad y_1 < x_1 < \cdots < y_N < x_N,$$

which will be denoted $x < y$ in the first case or $y < x$ in the second case.

Consider the Markov chain $(X(n))_{n\geq 0}$ in E_N starting from $X(0) = (x_1, \dots, x_N)$ where we obtain $X(n+1)$ from $X(n)$ by taking independently uniformly $X_k(n+1)$ in the Voronoï cell associated to $X_k(n)$, which is the interval $[\frac{X_{k-1}(n)+X_k(n)}{2}, \frac{X_k(n)+X_{k+1}(n)}{2}]$ i.e. the set of points in the circle which are the closer to $X_k(n)$. Let us denote by $m(X(n))$ the set of the boundaries of the Voronoï cells of $X(n)$. The coordinates of $m(X(n))$ are $\frac{X_k(n)+X_{k+1}(n)}{2}$ up to reordering and a shift so that they are in $[0, 1]$.

We remark that $m(X(n))$ and $X(n+1)$ are interlaced. Let us define the process Z_n^N valued in the integers that starts from 0 and increases by 1 if $m(X(n)) < X(n+1)$ and decreases by 1 in the other case. A first question could be to show that $\left(\frac{Z^N_{[nt]}}{\sqrt{n}}\right)_{0\leq t\leq 1}$ converges in distribution to $\sigma_N (B_t)_{0\leq t\leq 1}$ where B_t is a Brownian motion and $\sigma_N > 0$. The next step will be to compute σ_N in the spirit of [1] or at least $\lim_N \sigma_N$.

The interlacing property has been also considered in a different models of mathematical physics called Gefand-Tsetlin patterns. This topic could be developed into a larger project for a PhD student.

References

[1] Emmanuel Boissard, Serge Cohen, Thibault Espinasse, and James Norris. Diffusivity of a random walk on random walks. *Random Structures Algorithms*, 47(2):267–283, 2015.